

What drives AdS unstable?

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We calculate the spectrum of linear perturbations of standing wave solutions discussed in [Phys. Rev. D **87**, 123006 (2013)], as the first step to investigate the stability of globally regular, asymptotically AdS, time-periodic solutions discovered in [Phys. Rev. Lett. **111** 051102 (2013)]. We show that while this spectrum is only asymptotically nondispersive (as contrasted with the pure AdS case), putting a small standing wave solution on the top of AdS solution indeed prevents the turbulent instability. Thus we support the idea advocated in previous works that nondispersive character of the spectrum of linear perturbations of AdS space is crucial for the conjectured turbulent instability.

Introduction. A recent numerical and analytical study suggest that anti-de Sitter (AdS) space is unstable against the formation of a black hole under a large class of arbitrarily small perturbations [1–4]. It is argued [5, 6] that the two crucial ingredients of the mechanism of instability are: (1) the lack of dissipation of energy by radiation to null infinity (opposite to the Minkowski case) and (2) a resonant (nondispersive) spectrum of linear perturbations of AdS. This means that at linear level wave packets do not disperse in AdS and, once their modes get coupled through nonlinearities (coming either from self-gravity or self-interaction), it leaves a long time for them to interact. Then the (conserved) energy is efficiently transferred into higher and higher frequencies i.e. gets concentrated on finer and finer spatial scales that ultimately leads to a black hole formation. On the other hand it was suggested in [1, 4, 7] that there exist asymptotically AdS (aAdS) solutions that, being arbitrarily close to AdS, are immune to this instability. Indeed, two explicit examples of such stable stationary aAdS solutions were given, namely time-periodic solutions [8] and standing waves [9] (referred to as boson stars by the authors) for real and complex massless scalar field respectively. The existence of this type of solutions seems to be a rule rather than an exception. This suggests that while the AdS space itself is unstable against black hole formation, putting some fine-tuned small ripples on AdS can prevent the instability [7, 10]. But what makes these ripples stable? In this note we support the conclusions of the work [7] that a nondispersive character of the spectrum of linear perturbations on the fixed AdS background is crucial for the assumed instability. Firstly, up to now, there was some clash between analytical [7] and numerical results of one of us [11] to what extent only asymptotically resonant character of the spectrum is good enough to trigger the instability. We refined the numerical analysis of [11] and found that asymptotically resonant spectrum is not sufficient to trigger instability for small perturbations. Secondly, to investigate what makes the time-periodic solutions discovered in [8] stable we start with a simpler case of standing waves discussed in [9]. Namely, we investigate in detail the spectrum of their linear per-

turbations. We show that, while this spectrum is only asymptotically resonant, putting a small standing wave in the center of AdS prevents the instability.

Flat space enclosed in a cavity, revisited. In [11] a spherically symmetric self-gravitating massless scalar field enclosed in a perfectly reflecting spherical cavity was studied as a toy model for the assumed AdS instability. This somewhat artificial model allowed for two types of reflecting boundary conditions: Dirichlet and Neumann, resulting in strictly resonant (nondispersive) spectrum $\omega_j = j\pi/R$ and only asymptotically resonant spectrum $\tan R\omega_j = R\omega_j$, respectively (here R stands for the cavity radius). The resonant case, being a close analogue of the AdS case, showed a perfect scaling with the amplitude of the initial perturbation (compare the Fig. 2 in [11], with the key numerical evidence for AdS instability, the Fig. 2 in [1]) and the similar behavior of energy spectra to the AdS case (compare the Fig. 4 in [11] with the Fig. 2 in [6]) and strengthened the evidence for a robust mechanism of instability sketched in [1]. In spite of the fact that the analogous scaling in the Neumann boundary case (cf. Fig. 5 in [11]) might have not seem compelling enough, it was concluded in quest of further robustness that ... *the spectrum of linearized perturbations need not be fully resonant for triggering the instability.* On the other hand the authors of [7] came to the opposite conclusion based on nonlinear perturbation analysis. The clash between those two statements became even more prominent with the discovery of concrete examples of (nonlinearly) stable aAdS solutions [8, 9], previously advocated in [7] and the question what makes them immune to the instability discovered in [1]. Thus we ran the simulation for the same family of initial data as [11]

$$\Phi(0, r) = 0, \quad \Pi(0, r) = \varepsilon \exp\left(-64 \tan^2 \frac{\pi}{2} r\right), \quad (1)$$

but still smaller amplitudes [14]. The results are depicted in Fig. 1. For the Dirichlet boundary condition we confirmed the scaling depicted in the Fig. 2 of [11]: the scaling works better when the amplitude of the initial data is decreased. For the Neumann boundary condition we found that scaling does not improve as we decrease the

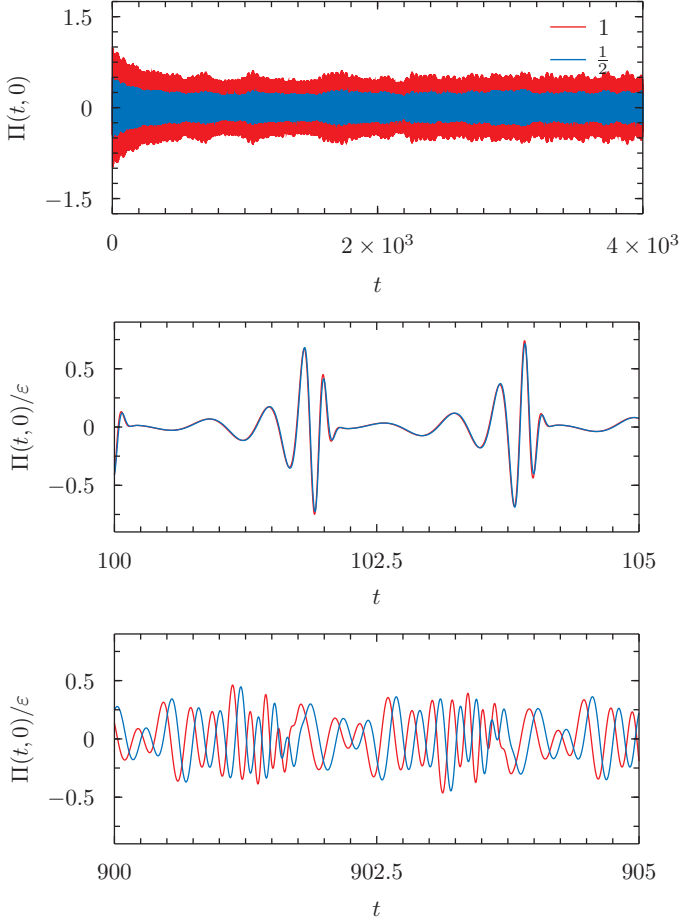


FIG. 1: *Top.* The function $\Pi(t, 0)$ for solutions to the model [11] with Neumann boundary condition at the cavity boundary (of size $R = 1$) with initial data (1) for small amplitude shows very different behaviour as opposed to moderate and large perturbations, see Fig. 5 in [11]. The spectral code [13] conserves the total mass up to 2.5×10^{-15} over long integration times. *Middle.* The closeup showing scaling of $\Pi(t, 0)$ function with an amplitude of the perturbation ε (with the same color coding). *Bottom.* Due to the dispersive spectra for Neumann boundary condition the initially localized perturbation spreads over the entire spatial domain which prevents the collapse. For late time there is also phase shift between the signals of different amplitudes.

amplitude, while for $\varepsilon \lesssim 1$ the instability is not triggered at all.

Standing waves in AdS. In this section we revisit the problem of nonlinear stability of standing waves (boson stars) in AdS and show that while the numerical results of [9] provide the evidence for their stability, the spectrum of their linear perturbation is only asymptotically resonant. To make this note self-contained we rewrite the equations for a complex, selfgravitating massless scalar field with negative cosmological constant studied extensively in [3, 9]. We parametrize the $(d + 1)$ -dimensional

asymptotically AdS metric by the ansatz

$$ds^2 = \frac{\ell^2}{\cos^2 x} (-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x d\Omega_{d-1}^2), \quad (2)$$

where $\ell^2 = -d(d-1)/(2\Lambda)$, $d\Omega_{d-1}^2$ is the round metric on S^{d-1} , $-\infty < t < \infty$, $0 \leq x < \pi/2$, and A, δ are functions of (t, x) . The evolution of the system is governed by Einstein equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \partial_\beta \bar{\phi} - \frac{1}{2} g_{\alpha\beta} \partial^\mu \phi \partial_\mu \bar{\phi} \right), \quad (3)$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0, \quad (4)$$

with ϕ standing for the massless complex scalar field. For the metric ansatz (2) this system boils down to (using the units with $8\pi G = d - 1$)

$$\dot{\Phi} = (Ae^{-\delta}\Pi)', \quad \ddot{\Pi} = \frac{1}{\tan^{d-1}x} (\tan^{d-1}x Ae^{-\delta}\Phi)', \quad (5)$$

$$\delta' = -\sin x \cos x (|\Phi|^2 + |\Pi|^2), \quad (6)$$

$$A' = \frac{d-2+2\sin^2 x}{\sin x \cos x} (1-A) + A\delta', \quad (7)$$

where $\dot{} = \partial_t$, $' = \partial_x$, and

$$\Phi = \phi', \quad \Pi = A^{-1}e^\delta \dot{\phi}. \quad (8)$$

Note that the set of equations (5-7) has the same form as in [2, 6, 8] with the only exception that auxiliary fields (8) are now complex valued functions, and it differs from one presented in [3] by the scaling factor $\cos^{d-1}x$. As discussed in [6] we supply this system with reflecting boundary conditions

$$\Pi(t, \pi/2) = 0, \quad \delta'(t, \pi/2) = 0, \quad A(t, \pi/2) = 1, \quad (9)$$

to require smooth evolution with a conserved total mass [6]. With the stationarity ansatz

$$\phi(t, x) = e^{i\Omega t} f(x), \quad \delta(t, x) = d(x), \quad A(t, x) = \mathcal{A}(x), \quad (10)$$

$\Omega > 0$, the system (5-8) is reduced to

$$-\Omega^2 \frac{e^d}{\mathcal{A}} f = \frac{1}{\tan^{d-1}x} (\tan^{d-1}x \mathcal{A} e^{-d} f')', \quad (11)$$

$$d' = -\sin x \cos x \left[f'^2 + \left(\frac{\Omega e^d}{\mathcal{A}} f \right)^2 \right], \quad (12)$$

$$\mathcal{A}' = \frac{d-2+2\sin^2 x}{\sin x \cos x} (1-\mathcal{A}) + \mathcal{A}d'. \quad (13)$$

We refer to solutions of the form (10) as standing wave solutions rather than boson star. For a review on a different models of boson star solutions and their possible astrophysical and cosmological relevance see [12] and references therein. We construct the solutions of the system (11-13) both numerically and perturbatively using a

modified versions of the codes for time-periodic solutions [8]. Using perturbative approach we seek for solution in a form

$$f(x) = \sum_{\text{odd } \lambda \geq 1} \varepsilon^\lambda f_\lambda(x), \quad f_1(x) = \frac{e_\gamma(x)}{e_\gamma(0)} \quad (14)$$

$$d(x) = \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda d_\lambda(x), \quad 1 - \mathcal{A}(x) = \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \mathcal{A}_\lambda(x), \quad (15)$$

$$\Omega = \omega_\gamma + \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \omega_{\gamma,\lambda}, \quad (16)$$

where $e_\gamma(x)$ is a dominant mode in the solution in the limit $\varepsilon \rightarrow 0$ ($e_j(x)$ is an eigenbasis of a linear problem for a fixed AdS_{d+1} background $Le_j(x) = \omega_j^2 e_j(x)$, $L = -\tan^{1-d} x \partial_x (\tan^{d-1} x \partial_x)$ with eigenfrequencies $\omega_j^2 = (d+2j)^2$, $j = 0, 1, \dots$; for explicit form of $e_j(x)$ see e.g. [8]). Since the $e_\gamma(x)$ function has exactly γ nodes we refer to the solution with dominant mode $\gamma = 0$ as a ground state solution while for solutions with $\gamma > 0$ as excited states (as in the boson star nomenclature). This particular choice of $f_1(x)$ together with a requirement $f_1(0) = 0$ for $\lambda \geq 3$ fixes a value of scalar field at the origin to $f(0) = \varepsilon$. Next, at each perturbative order λ we decompose scalar field $f_\lambda(x)$ and metric functions $\mathcal{A}_\lambda(x)$, $d_\lambda(x)$ in the eigenbasis $e_j(x)$ as for the time-periodic solutions, such that boundary conditions (9) are satisfied at each order. This allow us to obtain the solution by solving the linear algebraic system for Fourier coefficients, where the frequency corrections $\omega_{\gamma,\lambda}$ are fixed by an integrability conditions. In this way we get a unique solution, up to arbitrarily high order λ , with d and γ being the only parameters. For more details see [8].

As for the time-periodic solutions we also construct standing wave solution by solving (11-13) numerically. Here we represent the solution by a set of $3N$ Fourier coefficients

$$f(x) = \sum_{j=0}^{N-1} \hat{f}_j e_j(x), \quad d(x) = \sum_{j=0}^{N-1} \hat{d}_j (e_j(x) - e_j(0)), \quad (17)$$

$$\mathcal{A}(x) = 1 - \sum_{j=0}^{N-1} \hat{\mathcal{A}}_j e_j(x), \quad (18)$$

and the frequency Ω . With an approximation (17, 18) satisfying boundary conditions (9) we require for the equations (13-11) to be satisfied at the set of N collocation points. Additionally we add to this system a condition fixing the value of scalar field at the origin $f(0) = \varepsilon$. In this way we get a nonlinear eigenvalue system of $3N+1$ equations for the same number of unknowns approximating a solution to (11-13). Fig. 2 shows both the convergence rate of our numerical method and comparison with perturbatively constructed solution.

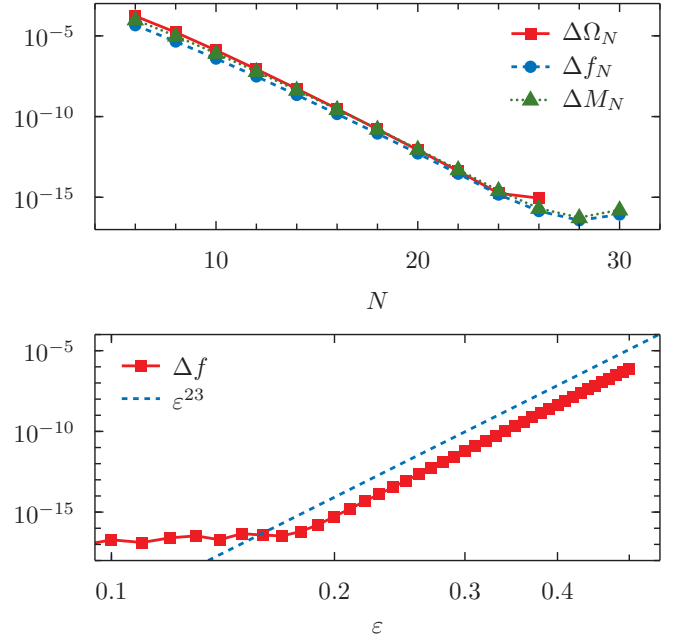


FIG. 2: *Top.* The convergence test of numerical code for ground state standing wave solution with $f(0) = 3/10$ ($\Omega \approx 4.56690$). The frequency error $\Delta\Omega_N = |\Omega_N - \Omega_{N=32}|$, scalar field profile error $\Delta f_N = \|f_N - f_{N=32}\|_2$, and total mass error $\Delta M_N = |M_N - M_{N=32}|$ computed for increasing number of Fourier coefficients N in (17, 18) compared with reference solution with $N = 32$. *Bottom.* The comparison of numerical and analytical ground state standing wave solutions for varying value of $f(0) = \varepsilon$. The scalar field absolute error $\Delta f = \|f_{\text{num}} - f_{\text{pert}}\|_2$ is computed for numerical solution with $N = 48$ eigenmodes, the perturbative series was found up to $\mathcal{O}(\varepsilon^{23})$ order. For small values of $\varepsilon < 0.2$ the rounding errors dominate. The discrete l^2 -norm $\|\cdot\|_2$ was computed on as set of equally spaced grid points $x_i = i\pi/800$, $i = 1, \dots, 400$.

Linear stability of standing waves. To study the linear stability we make the perturbative ansatz ($|\mu| \ll 1$)

$$\phi(t, x) = e^{i\Omega t} (f(x) + \mu \psi(t, x) + \mathcal{O}(\mu^2)), \quad (19)$$

$$\delta(t, x) = d(x) + \mu (\alpha(t, x) - \beta(t, x)) + \mathcal{O}(\mu^2), \quad (20)$$

$$A(t, x) = \mathcal{A}(x) (1 + \mu \alpha(t, x) + \mathcal{O}(\mu^2)), \quad (21)$$

$$(22)$$

and we neglect higher order terms in μ . Next, we assume harmonic time dependence of the perturbation

$$\psi(t, x) = \psi_+(x) e^{i\mathcal{X}t} + \psi_-(x) e^{-i\mathcal{X}t}, \quad (23)$$

$$\alpha(t, x) = \alpha(x) \cos \mathcal{X}t, \quad (24)$$

$$\beta(t, x) = \beta(x) \cos \mathcal{X}t, \quad (25)$$

where $\psi_+(x)$ and $\psi_-(x)$ are both real functions. This is the most general ansatz allowing for separation of t and x dependence, making at the same time the resulting system of equations relatively simple (cf. [9]). Plugging the (19-25) into (5-8) and linearizing about $\mu = 0$ we

obtain a set of differential-algebraic equations

$$\alpha = -\sin 2x \left\{ \frac{\Omega}{\mathcal{X}} f(\psi'_+ - \psi'_-) + f' \left[\left(1 - \frac{\Omega}{\mathcal{X}}\right) \psi_+ + \left(1 + \frac{\Omega}{\mathcal{X}}\right) \psi_- \right] \right\}, \quad (26)$$

$$\beta' = -\frac{d-1-\cos 2x}{\sin x \cos x} \frac{\alpha}{\mathcal{A}}, \quad (27)$$

$$\begin{aligned} \psi''_{\pm} = & -\frac{d-1-\cos 2x(1-\mathcal{A})}{\mathcal{A} \sin x \cos x} \psi'_{\pm} \\ & - \left(1 \mp \frac{\mathcal{X}}{\Omega}\right)^2 \left(\frac{\Omega e^d}{\mathcal{A}}\right)^2 \psi_{\pm} \\ & - \frac{1}{2} \beta' f' + \left(1 \mp \frac{\mathcal{X}}{2\Omega}\right) \left(\frac{\Omega e^d}{\mathcal{A}}\right)^2 \beta f. \end{aligned} \quad (28)$$

This system supplied with the boundary conditions (inherited from (9))

$$\psi_{\pm}(\pi/2) = 0, \quad \alpha(\pi/2) = 0, \quad \beta'(\pi/2) = 0, \quad (29)$$

and the regularity conditions at $x = 0$ is a linear eigenvalue problem with \mathcal{X} as an eigenvalue. In principle, knowing standing wave solution $f(x)$, $\mathcal{A}(x)$, $d(x)$, we could integrate (26-28) to obtain a solution in a closed form. Since this is not the case here, we again resort on perturbative method. We expand the unknown functions $\alpha(x)$, $\beta(x)$, $\psi_{\pm}(x)$ and frequency \mathcal{X} in small parameter ε (the same as for the standing wave solution (14-16))

$$\mathcal{X} = \sum_{\text{even } \lambda \geq 0} \varepsilon^{\lambda} \chi_{\lambda}, \quad \psi_{\pm}(x) = \sum_{\text{even } \lambda \geq 0} \varepsilon^{\lambda} \psi_{\pm, \lambda}(x), \quad (30)$$

$$\alpha(x) = \sum_{\text{odd } \lambda \geq 1} \varepsilon^{\lambda} \alpha_{\lambda}(x), \quad \beta(x) = \sum_{\text{odd } \lambda \geq 1} \varepsilon^{\lambda} \beta_{\lambda}(x). \quad (31)$$

Plugging (14-16) and (30-31) into (26-28) we demand that the equations are satisfied at each order of ε . Moreover, as for the standing wave solution we expand the unknown functions in eigenbasis $e_j(x)$

$$\psi_{\pm, \lambda}(x) = \sum_{j \geq 0} (e_j | \psi_{\pm, \lambda}) e_j(x), \quad (32)$$

$$\alpha_{\lambda}(x) = \sum_{j \geq 0} \hat{\alpha}_{\lambda, j} e_j(x), \quad (33)$$

$$\beta_{\lambda}(x) = \sum_{j \geq 0} \hat{\beta}_{\lambda, j} (e_j(x) - e_j(0)), \quad (34)$$

At the lowest order $\mathcal{O}(\varepsilon^0)$ the constraints (26) and (27) are identically satisfied, while from (28) we get two linear second order equations

$$L\psi_{\pm, 0} - (\chi_0 \mp \omega_{\gamma})^2 \psi_{\pm, 0} = 0, \quad (35)$$

Using decomposition of $\psi_{\pm, 0}(x)$ and orthogonality of the basis functions $(e_i | e_j) := \int_0^{\pi/2} e_i(x) e_j(x) \tan^{d-1} x dx = \delta_{i, j}$ we get the condition for the frequency χ_0

$$\begin{cases} \omega_j^2 - (\chi_0 - \omega_{\gamma})^2 = 0, \\ \omega_k^2 - (\chi_0 + \omega_{\gamma})^2 = 0. \end{cases} \quad (36)$$

This system is satisfied when: $\psi_{-, 0} \equiv 0$, $\psi_{+, 0} = e_j(x)$, and $\chi_0 = \omega_{\gamma} \pm \omega_j$ or $\psi_{+, 0} \equiv 0$, $\psi_{-, 0} = e_k(x)$, and $\chi_0 = -\omega_{\gamma} \pm \omega_k$ (there is also the case when neither of $\psi_{\pm, 0}(x)$ is zero, i.e. $\psi_{+, 0}(x) = e_j(x)$, $\psi_{-, 0}(x) = e_k(x)$ with k, j such that $d+2\gamma = |k-j|$ holds, but construction of solutions for this choice breaks down at higher orders, thus we exclude this case). Taking into account the form of the ansatz (25), due to the $t \rightarrow -t$ symmetry, these two seemingly different cases are in fact equivalent. Therefore, it suffices to consider the former case, so as a solution of the linear system (35) we take

$$\psi_{+, 0}(x) = e_{\zeta}(x), \quad \psi_{-, 0}(x) = 0, \quad \chi_0^{\pm} = \omega_{\gamma} \pm \omega_{\zeta}. \quad (37)$$

Thus, at the lowest order in ε , solution (37) specifies a standing wave with γ nodes perturbed by a single eigenmode with ζ nodes. Next, at each odd order λ the constraints are solved as follows. The coefficients $\hat{\alpha}_{\lambda, j}$ are simply given in terms of the decomposition of the order λ of the right hand side of the equation (26). Next, we rearrange the (27) at the order λ to obtain the linear system for the expansion coefficients of the $\beta_{\lambda}(x)$ function. For any even λ the system (26-28) reduces to two inhomogeneous equations

$$L\psi_{\pm, \lambda} - (\chi_0 \mp \omega_{\gamma})^2 \psi_{\pm, \lambda} = S_{\pm, \lambda}, \quad (38)$$

with source terms $S_{\pm, \lambda}$ depending on the lower order expansion coefficients in (14-16) and (30-31). Using the $\psi_{+, \lambda}(x)$ expansion formula (32) and projecting the first equation in (38) onto the $e_i(x)$ mode we have

$$(e_i | \psi_{+, \lambda}) = \frac{(e_i | S_{+, \lambda})}{\omega_i^2 - \omega_{\zeta}^2}, \quad i \neq \zeta, \quad (39)$$

where we have used the definition of χ_0^{\pm} given in (37). For $i = \zeta$ the necessary condition

$$(e_{\zeta} | S_{+, \lambda}) = 0, \quad (40)$$

is satisfied by an appropriate choice of the parameter χ_{λ} , while the free coefficient $(e_{\zeta} | \psi_{+, \lambda})$ is fixed as follows. We set the value of $\psi_{+}(x)$ at the origin to unity (we use the fact that governing equations are linear and we set $\psi_{+, 0}(x) = e_{\zeta}(x)/e_{\zeta}(0)$), then since $\psi_{+, 0}(0) = 1$ we require that $\psi_{+, \lambda}(0) = 0$ for $\lambda \geq 2$ which corresponds to taking

$$(e_{\zeta} | \psi_{+, \lambda}) = - \sum_{i \neq \zeta} (e_i | \psi_{+, \lambda}) e_i(0). \quad (41)$$

For a second equation in (38) after projection on $e_k(x)$ mode, we get

$$(e_k | \psi_{-, \lambda}) = \frac{(e_k | S_{-, \lambda})}{\omega_k^2 - (2\omega_\gamma \pm \omega_\zeta)^2}, \quad k \neq k_*, \quad (42)$$

where $\omega_{k_*} = |2\omega_\gamma \pm \omega_\zeta|$ and the sign depends on the particular choice of $\chi_0 = \chi_0^\pm$. For $\chi_0 = \chi_0^+ = \omega_\gamma + \omega_\zeta$ the $k_* = d + 2\gamma + \zeta > 0$ and the condition

$$(e_{k_*} | S_{-, \lambda}) = 0, \quad (43)$$

can always be satisfied by an appropriate choice of a constant $(e_{k_*} | \psi_{-, \lambda-2})$ (it is remarkable that at the lowest nontrivial order $\lambda = 2$ the coefficient $(e_{k_*} | S_{-, \lambda=2})$ is always zero for any combination of γ and ζ , so we can continue our construction to arbitrary high order λ , having exactly one undetermined constant after solving order λ , which will be fixed at higher order $\lambda + 2$). On the other hand for $\chi_0 = \chi_0^- = \omega_\gamma - \omega_\zeta$ we have $k_* = \frac{1}{2}(|d + 2(2\gamma - \zeta)| - d)$ which can be either positive or negative. For $k_* < 0$ there are always solutions to (42) since the denominator, on right hand side, is always different from zero for any $k \geq 0$, and the coefficient $(e_k | \psi_{-, \lambda})$ will be determined by the formula (42). The $k_* \geq 0$ case is more involved since there are two possibilities: either $d + 2(2\gamma - \zeta) \geq 0$ which gives $k_* = 2\gamma - \zeta \geq 0$ and there are no solutions to (42) since it turns out that the coefficient $(e_{k_*} | S_{2, \lambda=2})$ is nonzero, which leads to contradiction, either $d + 2(2\gamma - \zeta) < 0$ and

for $k_* = \zeta - 2\gamma - d \geq 0$ the coefficient $(e_{k_*} | S_{-, \lambda=2})$ is zero and the unknown $(e_{k_*} | \psi_{-, \lambda=2})$ will be fixed at higher order $\lambda = 4$ and we proceed just like for the $\chi_0 = \chi_0^+$ case. To sum up, for $\chi_0 = \chi_0^+$ there are solutions for any choice of γ and ζ , while for the $\chi_0 = \chi_0^-$ there exists solutions only for $\zeta > 2\gamma$.

In this way we construct a solution describing a standing wave with dominant eigenmode $e_\gamma(x)$ perturbed (at the linear level) by a dominant eigenmode $e_\zeta(x)$. Note the (general) ansatz (19-25) allows us to perturb a fixed standing wave with any eigenmode, as opposed to the analysis presented in [9]. The ansatz proposed in [9] restricts the form of perturbation, such that it allows for a γ -node standing wave to be perturbed by a solution with γ -nodes only. For that reason it is not suitable to find the full spectrum of linear perturbations.

Solving the higher orders of perturbative equations (in terms of ε expansion) we get successive approximation to the solution of the system (26-28) and in particular for the eigenfrequencies \mathcal{X} . Repeating this procedure for successive values of ζ we can compute the spectrum of linear perturbations around the standing wave (by deducing a general expression for frequency corrections χ_λ in perturbative series (30)). A systematic analysis of our results lead us to the observation that all of these corrections are given in terms of the recurrence relation which is easy to solve. Here we present just a sample of our calculations for ground state solution ($\gamma = 0$). For $\chi_0^+ = \omega_{\gamma=0} + \omega_\zeta$ the second order coefficient in (30) reads

$$\chi_2 = \frac{1134\zeta^6 + 19003\zeta^5 + 124820\zeta^4 + 407705\zeta^3 + 688426\zeta^2 + 548112\zeta + 146160}{448\zeta^5 + 5600\zeta^4 + 25760\zeta^3 + 53200\zeta^2 + 47292\zeta + 13230}, \quad (44)$$

for $\zeta = 0, 1, \dots$, while in the $\chi_0^- = \omega_{\gamma=0} - \omega_\zeta$ case we get

$$\chi_2 = \frac{-1134\zeta^6 - 8213\zeta^5 - 16920\zeta^4 - 455\zeta^3 + 28674\zeta^2 + 13168\zeta - 15120}{448\zeta^5 + 3360\zeta^4 + 7840\zeta^3 + 5040\zeta^2 - 1988\zeta - 1470}, \quad (45)$$

for $\zeta = 1, 2, \dots$. Having computed also higher order terms we can read off the asymptotic expansion of the linear spectrum of perturbed standing wave (30). Up to

fourth order in ε for large wave numbers ζ the spectrum (of ground state standing wave solution $\gamma = 0$) reads

$$\begin{aligned} \mathcal{X}^+ = & \left(2 + \frac{81\varepsilon^2}{32} + \frac{706663\varepsilon^4}{322560} + \dots\right) \zeta + \left(8 + \frac{1207\varepsilon^2}{112} + \frac{908257501\varepsilon^4}{86929920} + \dots\right) - \left(\frac{105\varepsilon^2}{64} + \frac{29319\varepsilon^4}{28672} + \dots\right) \zeta^{-1} \\ & + \left(\frac{165\varepsilon^2}{16} + \frac{472547\varepsilon^4}{28672} + \dots\right) \zeta^{-2} + \mathcal{O}(\zeta^{-3}), \end{aligned} \quad (46)$$

$$\begin{aligned} \mathcal{X}^- = & - \left(2 + \frac{81\varepsilon^2}{32} + \frac{706663\varepsilon^4}{322560} + \dots \right) \zeta + \left(\frac{73\varepsilon^2}{112} + \frac{48824929\varepsilon^4}{28976640} + \dots \right) + \left(\frac{105\varepsilon^2}{64} + \frac{29319\varepsilon^4}{28672} + \dots \right) \zeta^{-1} \\ & + \left(\frac{15\varepsilon^2}{4} + \frac{50753\varepsilon^4}{4096} + \dots \right) \zeta^{-2} + \mathcal{O}(\zeta^{-3}). \end{aligned} \quad (47)$$

Thus the spectrum is only asymptotically resonant for $\varepsilon \neq 0$. This has a direct consequence on the dynamics of perturbed standing wave solution, which we investigate in the next paragraph.

Numerical results. We solve the system (5-8) subject to boundary conditions (9) with the same methods as used in [8] with only minor modification due to real and imaginary parts of dynamical fields (8). For a purely real initial data

$$\begin{aligned} \Phi(0, x) &= 0, \\ \Pi(0, x) &= \varepsilon \frac{2}{\pi} \exp\left(-\frac{4 \tan^2 x}{\pi^2 \sigma^2}\right), \end{aligned} \quad (48)$$

(with $\sigma = 1/16$) we reproduce the scaling $\Pi(t, 0)^2 \rightarrow \varepsilon^{-2} \Pi(\varepsilon^2 t, 0)^2$ (cf. Fig. 2 in [1]) which improves with decreasing amplitude of the perturbation ε , supporting the conjectured AdS instability [1] for reflecting boundary conditions.

On the other hand for perturbed standing wave solution, i.e. for the initial data

$$\begin{aligned} \Phi(0, x) &= f'(x), \\ \Pi(0, x) &= \varepsilon \frac{2}{\pi} \exp\left(-\frac{4 \tan^2 x}{\pi^2 \sigma^2}\right) + i\Omega f(x) \frac{e^{d(x)}}{\mathcal{A}(x)}, \end{aligned} \quad (49)$$

(here we also set $\sigma = 1/16$) evolution is different (see Fig. 3 for a perturbed ground state $\gamma = 0$ solution; we observe the same behaviour for small amplitude excited states $\gamma > 0$). While for large amplitudes of the gaussian perturbation, after several dozens of reflections, the modulus of the scalar field $\Pi(t, 0)$ starts to grow indicating formation of apparent horizon, the situation changes when the perturbation becomes small. For slightly perturbed standing wave solution, and for simulated time intervals, the evolution does not show any sign of instability staying all the time close to the stationary state. Moreover, in contrast to perturbations of the pure AdS space, in this case we do not observe any scaling with the coordinate time t . Here we observe an initially narrow perturbation to bounce fourth and back over the standing wave solution, which as a time passes tends to spread out over the whole domain. This effect is a consequence of nonresonant spectra of standing wave solutions (46, 47), similarly to the Minkowski in a cavity model with Neumann boundary condition. The lack of a coherence restricts the energy transfers during successive implosions. As a consequence, the energy spectra of noncollapsing solution seems to equilibrate around some stationary distribution with small fluctuation of energy between eigenmodes (see Fig. 4). Therefore the solution behaves as a

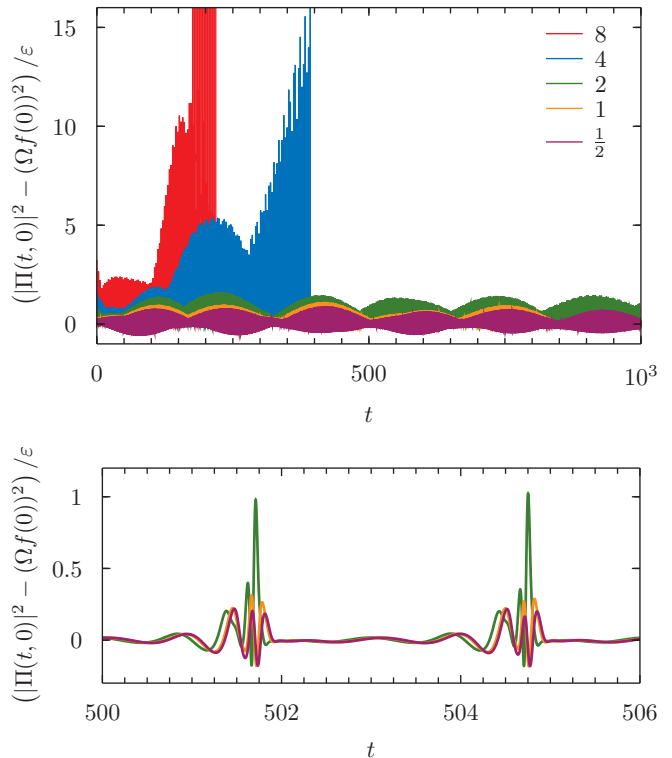


FIG. 3: *Top.* The time evolution (in $d = 4$) of squared modulus of a scalar field $\Pi(t, x)$ at the origin ($x = 0$) for a perturbed ground state standing wave solution with $f(0) = 0.16$ ($\Omega \approx 4.15034$) by narrow gaussian pulse (49) of decreasing amplitude (labelled by different line colors). *Bottom.* A closeup showing scaling with an amplitude of the perturbation ε , which improves when $\varepsilon \rightarrow 0$. Because of the nonlinearity of governing field equations we cannot exactly separate contributions coming from standing wave solution and a perturbation.

perturbation propagating on a standing wave background as is seen on Fig. 3 where we subtract a contribution of standing wave (a constant value) and rescale by the amplitude of initial perturbation.

Conclusions. There is growing evidence that while the AdS space is unstable against a black hole formation under a large class of arbitrarily small initial perturbations [1–4], there also exists a variety of stable, asymptotically AdS (aAdS) solutions like time-periodic solutions [8] or standing waves [9], that can be arbitrarily close to AdS. Both AdS and those aAdS solutions are stable at linear level. However, a small perturbations in a form of a short pulse of radiation, when perturbing the pure AdS solution, propagates roughly non-dispersively, with its energy cascading to higher frequencies and ultimately collapsing

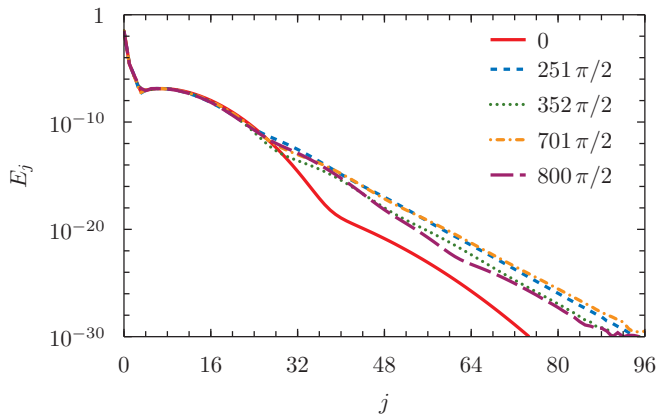


FIG. 4: Plot of the energy spectrum defined as $E_j := \left| \left(e_j \sqrt{A\Pi} \right) \right|^2 + \omega_j^{-2} \left| \left(e'_j \sqrt{A\Phi} \right) \right|^2$ at initial and late times (labelled by different line types) for the solution of perturbed standing wave (49) with $f(0) = 0.16$ ($\Omega \approx 4.15034$) and amplitude of the gaussian perturbation $\varepsilon = 1/2$. For late times spectrum falls off exponentially with an almost constant slope (compare with analogue Fig. 2 in [6] for perturbed AdS solution).

to a black hole, while perturbing those aAdS solutions (e.g. time-periodic ones), it disperses over the whole space and the energy cascade is ultimately stopped. It was suggested in [7] that this qualitative change in the long time evolution is due to the nonresonant character of the spectrum of linear perturbations of the aAdS solutions. In this note, as the first step to study the stability of time-periodic solutions, we investigate in detail the spectrum of linear perturbations of standing waves [9] and show that indeed it is not exactly resonant (it is only asymptotically resonant — resonant in the limit of a wave number going to infinity). Studying this system numerically we find that there is a threshold for triggering instability resulting in a black hole formation. It is important to stress that such stable aAdS solutions can be arbitrarily close to AdS. Then the fate of a small perturbations in a form of a short pulse of radiation depends on what dominates as perturbation of pure AdS: if “short pulse” dominates over a standing wave or a time-periodic solution (there is a “short pulse” perturbed with some small stable aAdS solution) it will still trigger the energy cascade and ultimately collapse to a black hole; if some stable aAdS solution dominates over a “short pulse” (there is a stable aAdS solution perturbed with a “short pulse”) the energy cascade is stopped and the evolution stays smooth. Then we confirm this qualitative behavior in the toy model of a portion of Minkowski space $\Lambda = 0$ enclosed in a perfectly reflecting cavity. The advantage of this somehow artificial model is that it allows for the two types of boundary conditions, resulting in either res-

onant or only asymptotically resonant spectrum of linear perturbations (for Dirichlet and Neumann boundary conditions respectively). Indeed the long time evolution of small perturbations for these two types of boundary conditions is qualitatively different. For the Dirichlet boundary conditions the perturbation ultimately collapses to a black hole (an analogue of the pure AdS case), while for Neumann boundary conditions (an analogue of a stable aAdS solution) there is a threshold for a black hole formation — the small perturbations do not collapse and their evolution stays smooth.

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